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Field Theory on Noncommutative Space-Time and the Deformed Virasoro Algebra

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Abstract

We consider a field theoretical model on the noncommutative cylinder which leads to a discrete-time evolution. Its Euclidean version is shown to be equivalent to a model on the complex q-plane. We reveal a direct link between the model on a noncommutative cylinder and the deformed Virasoro algebra constructed earlier on an abstract mathematical background. As it was shown, the deformed Virasoro generators necessarily carry a second index (in addition to the usual one), whose meaning, however, remained unknown. The present field theoretical approach allows one to ascribe a clear meaning to this second index: its origin is related to the noncommutativity of the underlying space-time. The problems with the supersymmetric extension of the model on a noncommutative super-space are briefly discussed.

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1 Introduction

The string theory and the noncommutative geometry are fundamental theories possessing common goals: the elimination of the problems appearing in the standard field theory, like ultra-violet divergences, and, perspectively, a quantization of the gravity. In the former theory the elementary objects are strings, i.e. they are not point-like, whereas from the latter the notion of a point, as the elementary geometrical or physical entity, is eliminated from the very beginning (for detailed discussion see books [1], [2] and [3]). However, it appears that there are deep relations between both approaches: the Yang-Mills theory on a noncommutative tori [4] can be re-interpreted as a limiting model of the M-theory [5].

In [6] the idea is discussed that the noncommutative space-time can appear naturally in the particular low energy limit of the string theory, as a direct consequence of a (constant) non-vanishing B-field. This fact is closely related to deformation quantization [7]. Conversely, the deformation quantization [8] can be interpreted in terms of a topological string theory [9]. Here the noncommutative geometry appears in the space-time (the target space of string coordinates).

In this article we shall analyze the different link between the string theory and the noncommutative geometry relating the known deformation of the Virasoro algebra with the noncommutative geometry on a string world-sheet.

The Virasoro algebra is an infinite Lie algebra with generators L_m, $m \in \mathbf{Z}$, satisfying commutation relations

$$[L_m, L_{m'}] \;=\; (m-m')L_{m+m'} \;+\; \frac{c}{12}(m^3-m)\delta_{m+m',0}\;, \tag{1}$$

where c is a central element commuting with all L_m. In an unitary irreducible representation $L_m^+ = L_{-m}$ and c is a real constant. The Virasoro algebra is usually realized in terms of an infinite set of (bosonic or fermionic) oscillators, and is closely related to the symmetry properties of the 2D (conformal) field theory in question. The simplest such representation, with $c = 1$, is given by the Sugawara construction of L_m in terms of an infinite set of bosonic oscillators a_n, $a_n^+ = a_{-n}$, $n = 1, 2, \ldots$,

satisfying the commutation relations

$$[a_n, a_m] \;=\; n\delta_{n+m,0} \ , \quad n, m \neq 0 \ . \tag{2}$$

The Sugawara formula for L_m reads

$$L_m \;=\; \frac{1}{2}\sum_{k,n\neq 0} : a_k a_n : \delta_{k+n,m} \ , \quad m \in \mathbf{Z} \ . \tag{3}$$

The deformations of the Virasoro algebra are related to deformations of the Sugawara construction (3). The oscillator realizations (with a nontrivial central element) of these deformations of Virasoro algebra were constructed in [10]. They have been intensively studied in [10]-[15], mainly in connection with a formal deformations in the conformal and/or string field theories. Later a developed mathematical structure was found, the Zamolodchikov-Faddeev algebras, which proved to be a natural framework for the deformed Virasoro algebras [16]. Our construction in [10] represents a paricular realization of a bosonization of a ZF algebra. For a recent review see [17].

Our strategy is as follows. First we describe a model on a standard (commutative) cylinder, in which framework the Virasoro algebra appears naturally. Then we generalize the model to the noncommutative analog of the cylinder. This leads to the discrete time evolution. In Sec. 2 we summarize briefly free scalar field theory on a commutative and noncommutative cylinder. In Sec. 3 we describe the Euclidean version of the model and analyze its symmetry properties. We show that there is a particular model on a noncommutative cylinder possessing, as a symmetry algebra, the deformed Virasoro algebra proposed earlier. Thus, the Virasoro algebra on a complex plane $\mathbf{C}$ is replaced by the deformed Virasoro algebra on a q-deformed complex plane $\mathbf{C}_q$: there is a direct link between the noncommutative geometry on a world-sheet and the deformation of the Virasoro algebra. This link allows us to ascribe a clear physical meaning to the second index of the deformed Virasoro generators which necessarily appears in all abstract generalizations of the Virasoro algebra. It turns out that appearance of the additional index is a straightforward consequence of the noncommutativity of coordinates of the space-time (q-complex plane or, equivalently, noncommutative cylinder) on which the underlying

field theory is defined. The field theoretical origin of the deformed Virosoro algebra can serve for the better (physical) motivation and understanding of its role as well as understanding of all related constructions (q-strings, q-vertex operators and Zamolodchikov-Faddeev algebras). Now all physical intuition (based on a notions like field action, quantization, Fock space, etc) can be used for further extensions, new constructions, etc originated and related to the q-deformed Virasoro algebras. In the last Sec. 4 we formulate the problem of a supersymmetric extension of the model and add concluding remarks.

2 Scalar field on a cylinder

2.1 Commutative case

In this section, we first discuss a free scalar field on a standard (commutative) cylinder C (see, e.g., [1], [2] and refs therein), and then we generalize the model to the noncommutative case. More detailed description of field-theoretical models on a noncommutative cylinder can be found in [18]-[20].

The cylinder C which we identify with the set of points $C = \mathbf{R} \times S^1 = \{x = (\rho \cos\phi, \rho \sin\phi, \tau) \in \mathbf{R}^3,\ \rho = \text{const}\}$. If the function $f(x)$ on C can be expanded as

$$f(\tau, \varphi) \;=\; \sum_{k \in \mathbf{Z}} c_k(\tau) e^{ik\varphi} \;, \tag{4}$$

we introduce the standard integral on C by putting

$$I_0[f] \;=\; \frac{1}{2\pi} \int_C d\tau d\varphi f(\tau, \varphi) \;=\; \int_{\mathbf{R}} d\tau \; c_0(\tau) \;. \tag{5}$$

The field action for a free massless real scalar field on a space-time modeled as the cylinder is defined by

$$S[\Phi] \;=\; \frac{1}{2} I_0[-\Phi \partial_\tau^2 \Phi - \Phi \partial_\varphi^2 \Phi] \;. \tag{6}$$

It can be interpreted as the action describing (one coordinate of the) free closed bosonic string. We shall have in mind this interpretation in what follows. We can

expand the fields Φ into Fourier modes

$$\Phi(x) \;=\; \sum_{k\neq 0} c_k(\tau)e^{ik\varphi} \;,\; c_k^*(\tau) \;=\; c_{-k}(\tau) \;, \tag{7}$$

(we eliminated the zero mode which is inessential for us). Inserting (7) into the action we obtain

$$S[\Phi] \;=\; \frac{1}{2}\int_{\mathbf{R}} d\tau \;\sum_{k\neq 0}[-c_{-k}(\tau)\ddot{c}_k(\tau) - k^2 c_{-k}(\tau)c_k(\tau)] \;. \tag{8}$$

The canonically conjugated momentum to the mode $c_k(\tau)$ is $\pi_k(\tau) = \dot{c}_{-k}(\tau)$. We assume standard equal-time canonical Poisson brackets between modes and their conjugate momenta

$$\{c_k(\tau), c_{k'}(\tau)\} \;=\; \{\pi_k(\tau), \pi_{k'}(\tau)\} \;=\; 0 \;,$$

$$\{c_k(\tau), \pi_{k'}(\tau)\} \;=\; \delta_{kk'} \;. \tag{9}$$

The quantization means an operator realization of the corresponding equal-time canonical commutation relations

$$[c_k(\tau), c_{k'}(\tau)] \;=\; [\pi_k(\tau), \pi_{k'}(\tau)] \;=\; 0 \;,$$

$$[c_k(\tau), \pi_{k'}(\tau)] \;=\; i\delta_{kk'} \;. \tag{10}$$

This can be performed by solving corresponding classical Euler-Lagrange equations of motion

$$-\;\ddot{c}_k(\tau) \;=\; k^2\, c_k(\tau) \;. \tag{11}$$

They possess positive frequency oscillating solutions

$$c_k(\tau) \;=\; \frac{i}{k}[a_k e^{-ik\tau} \;-\; b_{-k}e^{ik\tau}] \;,\; k > 0 \;. \tag{12}$$

The solutions for negative k are fixed by the reality condition: $c_{-k}(\tau) = c^* c_k(\tau)$, i.e. $a_{-k} = a_k^*$, $b_k^* = b_{-k}$. Explicitly, the formulas for the field $\Phi(\tau,\varphi)$ and the conjugated momentum $\Pi(\tau,\varphi) = \partial_\tau\Phi(\tau,\varphi)$ read

$$\Phi(\tau,\varphi) \;=\; \sum_{k\neq 0}\frac{i}{k}[a_k e^{-ik\tau+ik\varphi} \;-\; b_k e^{-ik\tau-ik\varphi}] \;,$$

$$\partial_\tau \Phi(\tau,\varphi) \;=\; \sum_{k\neq 0}[a_k e^{-ik\tau+ik\varphi} \;+\; b_k e^{-ik\tau-ik\varphi}]\;. \tag{13}$$

The terms with expansion coefficients a_k are interpreted as the right-movers on a closed bosonic string, whereas those solutions with b_k as the left-movers. They are independent, and we can treat them separately. The canonical commutations relations are indeed satisfied if we replace the complex coefficients a_k and b_k, $k \neq 0$, by two independent infinite set of bosonic oscillators satisfying commutation relations

$$[a_k, b_{k'}] \;=\; 0\;,\; [a_k, a_{k'}] \;=\; [b_k, b_{k'}] \;=\; k\delta_{k+k',0} \tag{14}$$

(we are using the same notation for the classical expansion parameters and annihilation and creation operators, this should not lead to any confusion).

2.2 Noncommutative case

The noncommutative cylinder we realize by "quantizing" the Poisson structure on C, see [18]-[20]. This is defined by

$$\{f,g\} \;=\; \frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial\tau} \;-\; \frac{\partial f}{\partial\tau}\frac{\partial g}{\partial\varphi}\;, \tag{15}$$

for any pair of functions f and g on C. Using the Leibniz rule, it can be generated from the elementary brackets

$$\{\tau, x_\pm\} \;=\; \pm i x_\pm\;,\; \{x_+, x_-\} \;=\; 0\;, \tag{16}$$

where, $x_\pm = \rho e^{\pm i\phi}$. Eqs. (17) are just $e(2)$ Lie algebra defining relations. The function x_+x_- is central: $\{x_0, x_+x_-\} = \{x_\pm, x_+x_-\} = 0$, i.e. the restriction $x_+x_- = \rho^2$ is consistent with the Poisson bracket structure. The operators ∂_φ^2 and ∂_τ^2 entering the action can be expressed in terms of Poisson brackets:

$$\partial_\varphi^2 f \;=\; \{\tau,\{\tau,f\}\}\;,\; \partial_\tau^2 f \;=\; \frac{1}{\rho^2}\{x_-,\{\{x_+,f\}\}\;. \tag{17}$$

In the noncommutative case we replace the commuting variables $\tau, x_\pm$ by the $e(2)$ Lie algebra generators $\hat\tau, \hat x_\pm$ satisfying the relations

$$[\hat\tau, \hat x_\pm] \;=\; \pm\lambda \hat x_\pm\;,\; [\hat x_+, \hat x_-] \;=\; 0\;,$$

$$\hat{x}_+ \hat{x}_- \;=\; \rho^2 \ . \tag{18}$$

The algebra $e(2)$ possesses one series of infinite dimensional unitary representations (parameterized by one real parameter $\rho > 0$). These representations can be realized in the Hilbert space $L^2(S^1, d\varphi)$ as follows

$$\hat{\tau} \;=\; -i\lambda\partial_\varphi \ , \ \hat{x}_\pm \;=\; \rho e^{\pm i\varphi} \ . \tag{19}$$

The Casimir operator takes the value $\hat{x}_+\hat{x}_- = \rho^2$. In what follows we put $\rho = 1$.

For any operator $\hat{f}$ on C possessing the expansion

$$\hat{f} \;=\; \sum_{k\in\mathbf{Z}} c_k(\hat{\tau}) e^{ik\varphi} \ , \tag{20}$$

we introduce the noncommutative analog of integral (5) by

$$I_\lambda[\hat{f}] \;=\; \lambda \mathrm{Tr}[\hat{f}] \;=\; \lambda \sum_{n\in\mathbf{Z}} c_0(n\lambda) \ . \tag{21}$$

Here $c_0(n\lambda)$ are spectral coefficients of the operator $c_0(\hat{\tau})$: $c_0(\hat{\tau})e^{in\varphi} = c_0(n\lambda)\ e^{in\varphi}$ ($e^{in\varphi}$ is an eigenfunction of $\hat{\tau}$ with the eigenvalue $n\lambda$). There is a straightforward generalization of (22) to an integral over finite discrete time interval $\alpha = a\lambda \le \tau \le b\lambda = \beta$:

$$I^\beta_{\lambda\alpha}[\hat{f}] \;:=\; \lambda \mathrm{Tr}^\beta_\alpha[\hat{f}] \;=\; \lambda \sum_a^b c_0(n\lambda) \ . \tag{22}$$

Integrals of this type appear, e.g., if one calculates the field action for a finite time interval.

The operators $\hat{\partial}^2_\varphi$ and $\hat{\partial}^2_\tau$, the noncommutative analogs of (18), are obtained replacing the Poisson brackets by the commutators: $\{.,.\} \to (i/\lambda)[.,.]$. Performing the Fourier expansion (21) we obtain

$$\begin{aligned} \hat{\partial}^2_\varphi \hat{f} \;&=\; -\frac{1}{\lambda^2}[\hat{\tau}, [\hat{\tau}, \hat{f}]] \;=\; -\sum_{k\in\mathbf{Z}} k^2 c_k(\hat{\tau}) e^{ik\varphi} \ , \\ \hat{\partial}^2_\tau \hat{f} \;&=\; -\frac{1}{\lambda^2\rho^2}[\hat{x}_-, [\hat{x}_+, \hat{f}]] \;=\; \sum_{k\in\mathbf{Z}} \delta^2_\lambda c_k(\hat{\tau}) e^{ik\varphi} \ . \end{aligned} \tag{23}$$

Here δ^2_λ is the second order difference operator

$$\delta^2_\lambda c_k(\hat{\tau}) \;=\; \frac{1}{\lambda^2}[c_k(\hat{\tau}+\lambda) - 2c_k(\hat{\tau}) + c_k(\hat{\tau}-\lambda)] \ .$$

The free hermitian scalar field action we take in the form

$$S[\hat{\Phi}] \;=\; \frac{1}{2} I_\lambda[-\Phi\partial_\tau^2\hat{\Phi} + K_\lambda^2(-i\partial_\varphi)\hat{\Phi}] \;, \tag{24}$$

where $K_\lambda(-i\partial_\varphi) = \frac{2}{\lambda}\sin(-i\frac{\lambda}{2}\partial_\varphi)$. This specific form of the operator K_λ has been chosen for the later convenience: as we shall see in section 3.2, the Euclidean version of (24) corresponds precisely to the action which can be reinterpreted as a theory on the noncommutative q-plane with a most simple and natural Lagrangian. The operator $K_\lambda(-i\partial_\varphi)$ is defined by the Fourier expansion (21): $K_\lambda(-i\partial_\varphi)\hat{f} = \sum c_k(\hat{\tau})K_\lambda(k)e^{ik\varphi}$; $K_\lambda(k) = \frac{2}{\lambda}\sin(\frac{k\lambda}{2})$.

The field $\hat{\Phi}$ is a function in the noncommutative variables $\hat{\tau}$ and $\hat{x}_\pm$. It possesses, by assumption, the Fourier expansion

$$\hat{\Phi} \;=\; \sum_{k\neq 0} c_k(\hat{\tau})e^{ik\varphi} \;, \quad c_k^*(\hat{\tau}) = c_{-k}(\hat{\tau} - k\lambda) \;. \tag{25}$$

The latter relation guarantees the hermiticity of the field:

$$\begin{aligned} \hat{\Phi}^* \;&=\; \sum_{k\neq 0} e^{-ik\varphi}c_k^*(\hat{\tau}) \;=\; \sum_{k\neq 0} c_k^*(\hat{\tau}+k\lambda)e^{-ik\varphi} \\ &=\; \sum_{k\neq 0} c_{-k}^*(\hat{\tau}-k\lambda)e^{ik\varphi} \;=\; \sum_{k\neq 0} c_k(\hat{\tau})e^{ik\varphi} \;=\; \hat{\Phi} \;. \end{aligned}$$

Inserting the mode expansion (25) into $S[\hat{\Phi}]$ and using the relation

$$c_k(\hat{\tau})e^{ik\varphi}c_{k'}(\hat{\tau})e^{ik'\varphi} \;=\; c_k(\hat{\tau})c_{k'}(\hat{\tau}-k\lambda)e^{i(k+k')\varphi} \;, \tag{26}$$

we obtain

$$S[\hat{\Phi}] \;=\; \frac{\lambda}{2}\sum_{n,k}[c_{-k}(n\lambda - k\lambda)\delta_\lambda^2 c_k(n\lambda) \;-\; K_\lambda^2(k)c_{-k}(n\lambda-k\lambda)c_k(n\lambda)] \;. \tag{27}$$

Its extremalization leads to the discrete-time Euler-Lagrange equations

$$-\,\delta_\lambda^2 c_k(n\lambda) \;=\; K_\lambda^2(k)c_k(n\lambda) \;. \tag{28}$$

We see that the noncommutativity demonstrates itself dominantly as a discreteness of time: the action (27) does not contain explicitely noncommutative quantities, it depends on spectral modes at various discrete time slices. Moreover, in the equations of motion the time derivatives are replaced by the time differences.

The momentum, conjugated to the field mode $c_k(n\lambda)$, is

$$\pi_k(n\lambda) \;=\; \frac{1}{2\lambda}[c_k(n\lambda - k\lambda + \lambda) - c_k(n\lambda - k\lambda - \lambda) \;.$$

We postulate the standard equal-time Poisson brackets among modes and conjugated momenta

$$\{c_k(n\lambda), c_{k'}(n\lambda)\} \;=\; \{\pi_k(n\lambda), \pi_{k'}(n\lambda)\} \;=\; 0 \;,$$
$$\{c_k(n\lambda), \pi_{k'}(n\lambda)\} \;=\; \delta_{kk'} \;. \tag{29}$$

The quantization means an operator realization of the corresponding equal-time canonical commutation relations

$$[c_k(n\lambda), c_{k'}(n\lambda)] \;=\; [\pi_k(n\lambda), \pi_{k'}(n\lambda)] \;=\; 0 \;,$$
$$[c_k(n\lambda), \pi_{k'}(n\lambda)] \;=\; i\delta_{kk'} \;, \;. \tag{30}$$

This can be performed similarly as in the commutative case, in terms of suitable sets of annihilation and creation operators. We shall not discuss this problem here since in the next Section, we analyze in detail the analogous problem within the Euclidean version.

3 Euclidean model on a cylinder

3.1 Commutative case

In order to analyze the Euclidean case (see, e.g., [1], [2]), we have to continue the time τ to $-i\tau$, as a result in the action the kinetic term changes sign. The Euclidean action is usually defined by

$$S[\Phi] \;=\; \frac{1}{2}I_0[-\Phi\partial_\tau^2\Phi - \Phi\partial_\varphi^2\Phi)] \;. \tag{31}$$

The corresponding Euler-Lagrange equations for modes

$$\ddot{c}_k(\tau) \;=\; k^2\, c_k(\tau) \;. \tag{32}$$

can be solved similarly as in the real-time case. The formula for the field $\Phi(\tau,\varphi)$ and the conjugate momentum $\Pi(\tau,\varphi)=\partial_\tau\Phi(\tau,\varphi)$ reads

$$\begin{aligned}\Phi(\tau,\varphi) &= \sum_{k\neq 0}\frac{i}{k}[a_k e^{-k\tau+ik\varphi} \ + \ b_k e^{-k\tau-ik\varphi}] \ ,\\ \partial_\tau\Phi(\tau,\varphi) &= -i\sum_{k\neq 0}[a_k e^{-k\tau+ik\varphi} \ + \ b_k e^{-k\tau-ik\varphi}] \ . \end{aligned} \tag{33}$$

The equal-time canonical commutation relations among $\Phi(\tau,\varphi)$ and $\Pi(\tau,\varphi')$ are satisfied provided that a_k and b_k, $k\neq 0$, satisfy commutation relations (14).

It is useful to introduce the new complex variables

$$z=e^{\tau-i\varphi}\ ,\ \bar z=e^{\tau+i\varphi}\ . \tag{34}$$

In this variables the action reads

$$S[\Phi] \ = \ \frac{1}{4\pi}\int dz d\bar z\,\partial\Phi(z,\bar z)\bar\partial\Phi(z,\bar z)\ . \tag{35}$$

The corresponding Euler-Lagrange equations

$$\partial\bar\partial\Phi(z,\bar z) \ = \ 0 \tag{36}$$

have a general solution

$$\Phi(z,\bar z) \ = \ i\sum_{k\neq 0}\frac{1}{k}[a_k z^{-k} \ + \ b_k\bar z^{-k}] =: \ \Phi(z) \ + \ \bar\Phi(\bar z)\ . \tag{37}$$

Comparing with (33), we see that the fields $\Phi(z)$ and $\bar\Phi(\bar z)$ correspond to the right- and left-movers, respectively, of a closed bosonic string.

The energy momentum tensor T_{ij} is traceless: $T_{z\bar z}=0$ in complex notation with $i,j=z,\bar z$. This, together with the energy-momentum conservation,

$$\bar\partial T_{zz}+\partial T_{\bar z z}=0\ ,\ \partial T_{\bar z\bar z}+\bar\partial T_{\bar z z}=0\ ,$$

gives $T(z,\bar z)=T(z) \ + \ \bar T(\bar z)$ with

$$T(z) \ = \ -\frac{1}{2}:\partial\Phi(z)\partial\Phi(z):\ ,\ \bar T(\bar z) \ = \ -\frac{1}{2}:\bar\partial\bar\Phi(\bar z)\bar\partial\bar\Phi(\bar z):\ . \tag{38}$$

Here there appears the normal product $:\ldots:$ defined by

$$:\Phi(z,\bar z)\Phi(w,\bar w):= \ \Phi(z,\bar z)\Phi(w,\bar w) \ + \ \log|z-w|^2\ . \tag{39}$$

The last term is just the 2-point correlator (Green function) $\langle\Phi(z,\bar{z})\Phi(w,\bar{w})\rangle$. It can be found, e.g. in the framework of Euclidean field theory, with the quantum expectation value of a field functional $F[\Phi]$ being given as the path integral

$$\langle F[\Phi]\rangle \ = \ \int D\Phi\, F[\Phi]\, e^{-S[\Phi]} \ . \tag{40}$$

Let us now consider the infinitesimal transformation of fields $\delta_\xi\Phi = \xi\partial\Phi$ with some given function $\xi = \xi(z,\bar{z})$. This induces the following variation of the action

$$\begin{aligned} \delta_\xi S \ &= \ \frac{1}{4\pi}\int dzd\bar{z}\,[(\bar\partial\xi)(\partial\Phi)^2 + \partial(\xi\,\partial\Phi\,\bar\partial\Phi)] \\ &= \ -\frac{1}{2\pi}\int dzd\bar{z}\,(\bar\partial\xi)T \ , \end{aligned} \tag{41}$$

with $T(z)$ given in (38) (here we used the formula $\int dzd\bar{z}\,\partial(\ldots) = 0$). The analogous formula for $\bar{T}(\bar{z})$ can be obtained by considering the variations $\delta_{\bar\xi}\Phi = \bar\xi\bar\partial\Phi$.

Inserting into $T(z)$ the mode expansion (33), using (38) and replacing the expansion coefficient by bosonic oscillators, we recover the Sugawara formula:

$$T(z) \ = \ \sum L_m z^{-m-2} \ , \tag{42}$$

where

$$L_m \ = \ \frac{1}{2\pi i}\oint z^{n+1}T(z) \ = \frac{1}{2}\sum :a_k a_n: \delta_{k+n,m} \ . \tag{43}$$

We note that here, on r.h.s., there appears the standard normal ordering with respect to the annihilation and creation operators. It can be shown that it is equivalent to the normal ordering introduced in (39), therefore, we have not introduced a new specific notation.

For variations $\delta_\xi\Phi = \xi\partial\Phi$ with $\xi = \xi(z)$ it holds $\delta_\xi S = 0$ (see (41)). Thus, they are symmetry transformations of the model. The variation $\delta_\xi\Phi$ in question corresponds to an infinitesimal conformal transformation $z \to z + \xi(z)$ of the conformal plane:

$$\Phi(z) \ \to \ \Phi(z+\xi(z)) \ = \ \Phi(z) + \xi(z)\partial\Phi(z) \ .$$

The infinitesimal conformal transformations of a plane form a Lie algebra, and the same is true for the variations: $[\delta_\xi, \delta_{\xi'}]\Phi = \delta_{\xi\partial_{\xi'} - \partial_\xi \xi'}\Phi$. Consequently, the Virasoro generators L_m, $m \in \mathbf{Z}$, close to a Lie algebra too. Wee see that the conformal mappings of a complex plane are behind the symmetry of the action in question.

3.2 Noncommutative case

Now we shall investigate along the similar lines the noncommutative Euclidean version of the model. It is obtained by replacing $\lambda \to -i\lambda$ in (27). The corresponding Euclidean action is

$$S[\hat{\Phi}] \;=\; \frac{1}{2} I_\lambda[-\hat{\Phi}\hat{\partial}_\tau^2\hat{\Phi} - \hat{\Phi}K_\lambda^2(-i\partial_\varphi)\hat{\Phi}] \;, \tag{44}$$

where now $K_\lambda(-i\partial_\varphi) = \frac{2}{\lambda}\sinh(-i\frac{\lambda}{2}\partial_\varphi)$. Inserting into (44) the field mode expansion, we obtain

$$S[\hat{\Phi}] \;=\; \frac{\lambda}{2}\sum_{n,k}[c_{-k}(n\lambda - k\lambda)\delta_\lambda^2 c_k(n\lambda) + K_\lambda^2(k)c_{-k}(n\lambda - k\lambda)c_k(n\lambda)] \;, \tag{45}$$

with $K_\lambda(k) = \frac{2}{\lambda}\sinh(\frac{k\lambda}{2})$. The corresponding discrete Euler-Lagrange equations

$$\delta_\lambda^2 c_k(n\lambda) \;=\; K_\lambda^2(k)c_k(n\lambda) \;,\; k \neq 0 \;, \tag{46}$$

have the solution

$$c_k(n\lambda) \;=\; \frac{i\lambda}{\sinh(k\lambda)}[a_k e^{\lambda k^2/2}e^{-kn\lambda} - b_{-k}e^{-\lambda k^2/2}e^{kn\lambda}] \;,\; k \neq 0 \;, \tag{47}$$

where a_k and b_{-k} are independent constants. The conjugate momentum to the mode $c_k(n\lambda)$ is

$$\begin{aligned} \pi_k(n\lambda) \;&=\; \frac{1}{2\lambda}[c_{-k}(n\lambda - k\lambda - \lambda) - c_{-k}(n\lambda - k\lambda + \lambda)] \\ &=\; [a_{-k}e^{-\lambda k^2/2}e^{kn\lambda} + b_k e^{\lambda k^2/2}e^{-kn\lambda}] \;,\; k \neq 0 \;. \end{aligned} \tag{48}$$

The Euclidean reality condition, $\hat{\Phi}^\dagger(\hat{\tau},\varphi) \stackrel{\text{def}}{=} \hat{\Phi}^*(-\hat{\tau},\varphi) = \hat{\Phi}(\hat{\tau},\varphi)$, requires $c_k(n\lambda) = c_{-k}^*(-n\lambda + k\lambda)$. Taking this into account, the canonical commutation relations (30) are satisfied provided that the coefficients a_k and b_k, $k \neq 0$, are replaced by bosonic operators satisfying the commutation relations

$$[a_k, b_{k'}] \;=\; 0 \;,\; [a_k, a_{k'}] \;=\; [b_k, b_{k'}] \;=\; \frac{\sinh(k\lambda)}{\lambda}\delta_{k+k',0} \;. \tag{49}$$

Inserting solution (47) into the field mode expansion, we obtain the field configuration, minimizing the action, in the form

$$\hat{\Phi}(\hat{z},\hat{\bar{z}}) \;=\; \sum_{k\neq 0}\frac{i\lambda}{\sinh(k\lambda)}[a_k\hat{z}^{-k} - b_k\hat{\bar{z}}^{-k}] \;=:\; \hat{\Phi}(\hat{z}) \;+\; \hat{\bar{\Phi}}(\hat{\bar{z}}) \;. \tag{50}$$

Here we introduced the new noncommutative variables

$$\hat{z} \ = \ e^{\hat{\tau}-i\varphi} \ = \ e^{-i\lambda\partial_\varphi - i\varphi} \ , \ \hat{\bar{z}} = \ e^{\hat{\tau}+i\varphi} \ = \ e^{-i\lambda\partial_\varphi + i\varphi} \ . \tag{51}$$

The operators $\hat{z}$ and $\hat{\bar{z}}$ satisfy the commutation relation

$$\hat{z}\hat{\bar{z}} \ = \ q^2\hat{\bar{z}}\hat{z} \ , \ q = e^\lambda \ . \tag{52}$$

which are typical for the q-deformed plane $\mathbf{C}_q$ possessing an $E_q(2)$ symmetry. Bellow we summarize necessary formulas following the conventions of [21].

The q-deformed Euclidean group $E_q(2)$ is a Hopf algebra generated by v, $\bar{v}$, n, $\bar{n}$ with relations

$$vn \ = \ q^2 nv \ , \ v\bar{n} \ = \ q^2\bar{n}v \ , \ n\bar{n} \ = \ q^2\bar{n}n \ ,$$

$$\bar{n}\bar{v} \ = \ q^2\bar{v}\bar{n} \ , \ n\bar{v} \ = \ q^2\bar{v}n \ = \ v\bar{v} \ = \ \bar{v}v \ = \ 1 \ ,$$

with comultiplication, counit and antipode given by

$$\Delta v \ = \ v\otimes v \ , \ \Delta\bar{v} \ = \ \bar{v}\otimes\bar{v} \ , \ \Delta n \ = \ n\otimes 1 + v\otimes n \ , \ \Delta\bar{n} \ = \ \bar{n}\otimes 1 + \bar{v}\otimes\bar{n} \ ,$$

$$\varepsilon(v) = \varepsilon(\bar{v}) = 1 \ , \ \varepsilon(n) = \varepsilon(\bar{n}) = 0 \ ,$$

$$S(v) = \bar{v} \ , \ S(\bar{v}) = v \ , \ S(n) = -\bar{v}n \ , \ S(\bar{n}) \ = \ -v\bar{n} \ .$$

For a real q the compatible involution is $v^* = \bar{v}$, $n^* = \bar{n}$.

The dual enveloping algebra $\mathcal{U}_q(e(2))$ is generated by the elements $P_\pm$ and J such that

$$[P_+, P_-] = 0 \ , \ [J, P_\pm] = \pm P_\pm \ ,$$

with vanishing counit and

$$\Delta J = J\otimes 1 + 1\otimes J \ , \ \Delta P_\pm = q^{-J}\otimes P_\pm + P_\pm\otimes q^J \ ,$$

$$S(J) = -J \ , \ S(P_\pm) = -q^{\pm 1}P_\pm \ , \ J^* = J \ , \ P_\pm^* = P_\mp \ .$$

The plane $\mathbf{C}_q$ is associated with functions analytic in $\hat{z}$, $\hat{\bar{z}}$. It is invariant with respect to the $\mathcal{U}_q(e(2))$ coaction Λ given by (see [21]):

$$\Lambda(q^{\pm J})\hat{z}^m\hat{\bar{z}}^n = q^{\pm m\mp n}\hat{z}^m\hat{\bar{z}}^n \ ,$$

$$\Lambda(P_-)\hat{z}^m\hat{\bar{z}}^n \equiv \partial_q \hat{z}^m\hat{\bar{z}}^n = [m]_q q^{n-2}\hat{z}^{m-1}\hat{\bar{z}}^n \ ,$$

$$\Lambda(P_+)\hat{z}^m\hat{\bar{z}}^n \equiv \bar{\partial}_q \hat{z}^m\hat{\bar{z}}^n = -[n]_q q^{m+1}\hat{z}^m\hat{\bar{z}}^{n-1} \ ,$$

where $[m]_q = (q^m - q^{-m})/(q-q^{-1})$. The corresponding Casimir operator (the Laplacian on $\mathbf{C}_q$) is $\Delta_q = \Lambda(P_+P_-)$. Its action on $\mathbf{C}_q$ is

$$\Delta_q \hat{z}^m\hat{\bar{z}}^n = \Lambda(P_+P_-)\hat{z}^m\hat{\bar{z}}^n = -q^{m+n-2}[m]_q[n]_q\hat{z}^{m-1}\hat{\bar{z}}^{n-1} \ . \tag{53}$$

The operator $\Delta_\lambda = -\hat{\partial}_\tau^2 + \frac{4}{\lambda}^2 \sinh^2(-i\frac{\lambda}{2}\partial_\varphi)$ entering (44) acts on $\hat{z}^m\hat{\bar{z}}^n = e^{(m+n)\hat{\tau}-i(m-n)\varphi-mn\lambda}$ as follows

$$\begin{aligned}\Delta_\lambda \hat{z}^m\hat{\bar{z}}^n &= [-\hat{\partial}_\tau^2 + \frac{4}{\lambda}^2 \sinh^2(-i\frac{\lambda}{2}\partial_\varphi)]\hat{z}^m\hat{\bar{z}}^n \\ &= -\frac{(q-q^{-1})^2}{\lambda^2}[m]_q[n]_q\hat{z}^m\hat{\bar{z}}^n \ , \ q = e^\lambda \ .\end{aligned} \tag{54}$$

Comparing (53) and (54) it can be shown straightforwardly that

$$\Delta_\lambda \hat{z}^m\hat{\bar{z}}^n \ = \ \frac{(q-q^{-1})^2}{\lambda^2}\hat{r}(\Delta_q \hat{z}^m\hat{\bar{z}}^n)\hat{r} \ , \ \hat{r} = e^{\hat{\tau}} \ . \tag{55}$$

This is the noncommutative analog of the known link between Laplacian on a plane and those on a cylinder.

To any function $f(\hat{z},\hat{\bar{z}}) = \sum_{n,m\geq 0} C_{m,n}\hat{z}^m\hat{\bar{z}}^n$ on $\mathbf{C}_q$ we assign the Jackson-type integral by

$$\int_q d\hat{z}d\hat{\bar{z}}\, f(\hat{z},\hat{\bar{z}}) \ = \ \lambda \mathrm{Tr}[\hat{r}^2 f_0(\hat{r}^2)] \ = \ \lambda \sum_{n\in\mathbf{Z}} q^{2n} f_0(q^{2n}) \ , \tag{56}$$

where Tr denotes the trace over the spectrum of $\hat{r}^2$ is q^{2n}, $n \in \mathbf{Z}$, and

$$f_0(\hat{r}) \ = \ \sum_{n\geq 0} C_{n,n} q^{n^2}\hat{r}^{2n} \ . \tag{57}$$

We can extend the definition (56) by taking a partial trace over the spectrum: putting $\alpha = q^{2a}$ and $q^b = \beta$, we define

$$\int_{q\alpha}^{\beta} d\hat{z}d\hat{\bar{z}}\, f(\hat{z},\hat{\bar{z}}) \ = \ \lambda \sum_{n=a}^{b} q^{2n} f_0(q^{2n}) \ . \tag{58}$$

Taking $\hat{f} = \hat{r}^2 f(\hat{z},\hat{\bar{z}})$ it can be seen easily that the integrals $I_\lambda[\hat{f}]$ and $\int_q d\hat{z}d\hat{\bar{z}}\, f(\hat{z},\hat{\bar{z}})$ are equal. The factor $\hat{r}^2$ represents a Jacobian of the transformation (51). We see

that in the noncommutative case there are two equivalent approaches, linked by the transformation (51):

(i) The first one corresponds to the model on a noncommutative sphere described by the noncommutative variables $\hat{r}$ and φ. The field action (44) leads to the equations of motion (46).

(ii) The second one is a model on a q-plane described by the variables $\hat{z}$, $\hat{\bar{z}}$. The action

$$S[\hat{\Phi}] \; = \; \int_q d\hat{z}d\hat{\bar{z}} \; \hat{\Phi}(\hat{z}q, \hat{\bar{z}}q^{-1})(-\Delta_q)\hat{\Phi}(\hat{z}, \hat{\bar{z}}) \; . \tag{59}$$

The shifts of arguments in the first $\hat{\Phi}$ guarantee that (59) is equivalent to (44) (use the link (53) between Laplacians and the relation $\hat{r}\hat{\Phi}(\hat{z}q, \hat{\bar{z}}q^{-1}) = \hat{\Phi}(\hat{z}, \hat{\bar{z}})\hat{r}$).

The action (59) depends on general hermitian field configurations

$$\hat{\Phi}(\hat{z}, \hat{\bar{z}}) \; = \; \sum_{k\neq 0} \frac{i\lambda}{\sinh(k\lambda)} [a_k(q^k\hat{r}^2)\hat{z}^{-k} - b_{-k}(q^{-k}\hat{r}^2)\hat{\bar{z}}^{-k}] \; . \tag{60}$$

The field is hermitian provided that $a^*_{-k}(q^k\hat{r}^{-2}) = a_k(q^{-1}\hat{r}^2)$. The Euler-Lagrange equation for $S[\hat{\Phi}]$ is just the q-harmonicity condition

$$\Delta_q\hat{\Phi}(\hat{z}, \hat{\bar{z}}) \; = \; \partial_q\bar{\partial}_q\hat{\Phi}(\hat{z}, \hat{\bar{z}}) \; = \; 0 \; . \tag{61}$$

Obviously, it is equivalent to the equation of motion (46).

Let us now investigate the variations of the action under infinitesimal field transformations

$$\delta_{\hat{\xi}}\hat{\Phi} \; = \; \hat{\xi}\partial_q\hat{\Phi} \; , \; \hat{\xi} = \xi(\hat{z}, \hat{\bar{z}}) \; . \tag{62}$$

Using the Leibniz rule

$$\partial_q[f(\hat{z}q, \hat{\bar{z}}q)\, g(\hat{z}, \hat{\bar{z}})] \; = \; (\partial_q f)(\hat{z}q, \hat{\bar{z}}q)\, g(\hat{z}q, \hat{\bar{z}}q) \; + \; f(\hat{z}q, \hat{\bar{z}}q)\, (\partial_q g)(\hat{z}q, \hat{\bar{z}}q) \; ,$$

we can rewrite the action in the form

$$S[\hat{\Phi}] \; = \; q^2 \int_q d\hat{z}d\hat{\bar{z}} \; (\partial_q\hat{\Phi})(\hat{z}q, \hat{\bar{z}}q^{-1})\, (\bar{\partial}_q\hat{\Phi})(\hat{z}, \hat{\bar{z}}) \; .$$

It follows straightforwardly,

$$\delta_{\hat{\xi}}S[\hat{\Phi}] \; = \; q^2 \int_q d\hat{z}d\hat{\bar{z}} \; [\partial_q(\hat{\xi}\partial_q\hat{\Phi})(\hat{z}q, \hat{\bar{z}}q^{-1})\, (\bar{\partial}_q\hat{\Phi})(\hat{z}, \hat{\bar{z}}) \; ,$$

$$+ \ (\partial_q\hat{\Phi})(\hat{z}q,\hat{\bar{z}}q^{-1})\,\bar{\partial}_q(\hat{\xi}\partial_q\hat{\Phi})(\hat{z},\hat{\bar{z}})] \ . \tag{63}$$

We shall now restrict ourselves to the vicinity of solutions of the equation of motion, i.e. we take

$$\hat{\Phi}(\hat{z},\hat{\bar{z}}) \ = \ \Phi(\hat{z}) \ + \ \bar{\Phi}(\hat{\bar{z}}) \ .$$

The first term in (63) does not contribute since, the integrand can be written as $\partial_q(\hat{\xi}\partial_q\Phi\bar{\partial}_q\bar{\Phi})$ and $\int_q \partial_q(\ldots) = 0$. Thus,

$$\delta_{\hat{\xi}}S[\hat{\Phi}] \ = \ q^2\int_q d\hat{z}d\hat{\bar{z}}\ (\partial_q\Phi)(\hat{z}q,\hat{\bar{z}}q^{-1})\,(\bar{\partial}_q)\xi(\hat{z},\hat{\bar{z}})\,(\partial_q\Phi(\hat{z},\hat{\bar{z}}) \ . \tag{64}$$

Any function $\hat{\xi} = \xi(\hat{z},\hat{\bar{z}})$ can be written as a linear combination of functions $\hat{\xi}_k = \eta(\hat{z})\hat{\bar{z}}^k$. It holds

$$\delta_{\hat{\xi}_k}S[\hat{\Phi}] \ = \ \int_q d\hat{z}d\hat{\bar{z}}\ (\partial_q\Phi)(\hat{z}q)\,(\bar{\partial}_q\hat{\xi}_k)(\hat{z},\hat{\bar{z}})\,(\partial_q\Phi)(\hat{z})$$

$$= \ q^2\int_q d\hat{z}d\hat{\bar{z}}\ (\bar{\partial}_q\hat{\xi}_k)(\hat{z},\hat{\bar{z}})\,(\partial_q\Phi)(\hat{z}q^{2k+1})\,(\partial_q\Phi)(\hat{z}) \ .$$

Shifting $\hat{z} \to \hat{z}q^{-k-\frac{1}{2}}$ we obtain for any k the generator of field transformation in the splitted form

$$T_k(\hat{z}) \ = \ -q^2 : (\partial_q\Phi)(\hat{z}q^{k+\frac{1}{2}})\,(\partial_q\Phi)(\hat{z}q^{-k-\frac{1}{2}}) \ . \tag{65}$$

This is exactly the formula for the splitted Virasoro generators, [14]-[16]. Inserting here the mode expansion (see (50))

$$(\partial_q\Phi)(\hat{z}) \ = \ \frac{i\lambda q^{-2}}{(q-q^{-1})}\sum_{k\neq 0} a_k\hat{z}^{-k-1} \ , \tag{66}$$

we obtain

$$T_k(\hat{z}) \ = \ -\frac{\lambda^2}{(q^2-1)^2}\sum_{n\in\mathbf{Z}} L_n^k\hat{z}^{-n-2} \ . \tag{67}$$

Here

$$L_n^k \ = \ \sum_{l,l'\neq 0} q^{(k-1)(l-l')} : a_l a_{l'} : \delta_{l+l',n} \ , \tag{68}$$

are generators of the (double indexed) deformed Virasoro algebra proposed in [10]:

$$[L_n^k, L_{n'}^{k'}] \ = \ \frac{1}{4}\sum_{\sigma\sigma'}[\frac{n-n'}{2} + n\sigma'k' + n'\sigma k]_- L_{n+n'}^{\sigma k'-\sigma' k} \ + \ C_n^{kk'}\delta_{n+n',0} \ , \tag{69}$$

where

$$C_n^{kk'} = \frac{1}{2}\sum_{m=1}^{n}[(n-2m)k]_+[(n-2m)k']_+[m]_-[n-m]_- \ . \tag{70}$$

Here we introduced the notation $[x]_- = (q^x - q^{-x})/(q - q^{-1})$ and $[x]_+ = (q^x + q^{-x})/2$.

Some comments are in order:

(i) In the commutative case the formula (42) for the transformation generators $T(z)$ follows for a general $\xi(z,\bar{z})$ and a general field configuration. In addition, in the commutative version, the model has the conformal symmetry.

(ii) In the noncommutative case the situation is different: we have different expressions for $T_k(\hat{z})$ for integers k (depending on the point splitting of the arguments in (65)) which generate transformations of q-harmonic functions among each other. They form the deformed Virasoro algebra introduced in [10] (a particular realization of the Zamolodchikov-Faddeev algebra [16]). Notice that the appearance of the additional index k in the deformed Virasoro algebra is directly related to the noncommutativity of the underlying two-dimensional space-time.

(iii) There are various reasons for which these symmetries of the model can not have as a background some "conformal symmetry" of a q-plane: 1) transformations of the type $z \to z + \xi(z)$ do not spoil the $\mathbf{C}_q$ structure only for a very limited set of $\xi(z)$, and 2) the simple q-Taylor expansion formula $\Phi(z+\xi(z)) = \Phi(z) + \xi(z)\partial_q\Phi(z) + \ldots$, is not valid even for an infinitesimal $\xi(z)$. The meaning of symmetry transformations, generated by the deformed Virasoro operators, requires further investigations. This problem is currently under study.

4 Concluding remarks

Recently have been found deep relations between string theory and noncommutative geometry in the space-time [6], [7]. We have analyzed an alternative possibility introducing the noncommutative geometry on a string world-sheet. To achieve this aim we have investigated a free bosonic string on a noncommutative cylinder. Our results can be summarized as follows:

- The field theory on a noncommutative cylinder leads in a natural way to the

discrete time evolution [18]-[20]. We started with a suitable model for a free scalar field (the one component of bosonic string) on a noncommutative cylinder with a suitable particular symmetry of the field action.

- The model in the Euclidean version can be equivalently formulated as a model on a q-deformed complex plane $\mathbf{C}_q$. Its symmetry is described by the deformed Virasoro algebra suggested earlier [10], appearing in the context of Zamolodchikov-Faddeev algebras [16].

The field theoretical origin of the deformed Virosoro algebra can serve for a better (physical) motivation and understanding of its role in all related constructions (q-strings, q-vertex operators and Zamolodchikov-Faddeev algebras). We see that the suggested formal deformation of the Virasoro algebra appears not only within the developed mathematical structure of Zamolodchikov-Faddeev algebras but also that there exists a physical theory, namely, the free bosonic string on a noncommutative cylinder, in which framework the deformed Virasoro algebra emerges as the symmetry of the theory. This fact allows one to justify the appearance and to clarify the meaning of the second index which labels (in addition to the usual one) the deformed Virasoro generators. As we have shown, this is related to the noncommutativity of the underlying two-dimensional space (world-sheet).

In this context it would be of great interest to extend our model to the supersymmetric case. We have strong indications that this can be achieved along the same lines as in the bosonic case:

(i) One can start from the fermionic realization of the deformed Virasoro algebra (69) proposed in [11]. Introducing the Dirac operator on a noncommutative cylinder this can be interpreted in terms of a fermionic spinor field on a noncommutative cylinder.

(ii) Such spinor model can be reformulated as a theory on a noncommutative supercylinder. Consequently, the bosonic and fermionic realizations can be joined to a superfield theory on a noncommutative supercylinder. In the Euclidean version the resulting theory is formulated on a q-deformed superplane (with symmetries described by the quantum supergroup s-$E_q(2)$).

Both indicated steps require careful constructions of all noncommutative analogs of objects in question (the noncommutative (super)cylinder, Dirac operator, operator orderings, etc). Investigations in this direction are under current study.

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